Math 33A

HW 2 Due 7/15 11:59 pm

OH: Thurs. changed to 3-4 pm

Today: Image, Kernel, Bases, Orthogonality

Recap : For any matrix A :

$$\operatorname{Im}(A) = \{\text{span of the columns of } A\}$$

$$\operatorname{Ker}(A) = \{\text{All the solutions to } A\vec{x} = \vec{0}\}$$

Subspaces : Collections of vectors closed under scalar multiplication + vector addition

$\Leftrightarrow$

Spans : $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$

all linear combinations of $\vec{v}_1, \dots, \vec{v}_n$

Basis: A basis for a subspace V is a collection of **linearly independent** vectors $\vec{b}_1, \ldots, \vec{b}_n$ s.t.

$$V = \text{span}\{\vec{b}_1, \ldots, \vec{b}_n\}$$

$$\left(c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n = \vec{0} \iff c_1 = c_2 = \cdots = c_n = 0 \right)$$

The size of a basis for a subspace is the dimension of that subspace.

Ex:

1.1: $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix}$ Find a basis for ker A & a basis for im A

Algorithm for finding a basis for im(A) & ker(A)

① Row reduce A until we find all the leading 1's

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix} \begin{matrix} \\ -2(I) \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

② Every col. w/ a leading 1 corresponds to a basis vector for im(A) in the original matrix

$$\text{im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

(3) To find ker(A), augment row reduced matrix w/ the 0 vector (& identify any free variables)

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_3, x_4 free variables

(4) Use matrix to write non-free vars. in terms of the free variables

$$x_1 + 2x_2 - x_3 + 3x_4 = 0 \implies x_1 = -2x_2 + x_3 - 3x_4$$

(5) Plug-in my eqns. for my non-free variables into the vector form of my solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 - 3x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(6) Break apart my sol. vector by component

$$\begin{bmatrix} -2x_2 + x_3 - 3x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

7. The basis for ker(A) is given by each of my final vectors

$$\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\underset{\text{(\# of leading 1's)}}{\dim(\operatorname{im}(A))} + \underset{\text{(\# of free variables)}}{\dim(\ker(A))} = \text{\# of cols of } A \quad \text{(ALWAYS TRUE)}$$

(Rank-nullity thm)

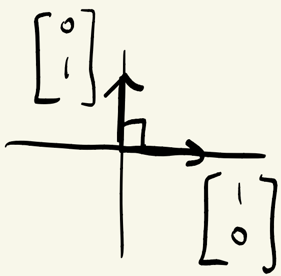
Orthogonality :

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

Dot product : $\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$

$\vec{v}, \vec{w}$ are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$

ex :



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \cdot 0 + 0 \cdot 1 = 0 + 0 = 0$$

Orthogonal Projection:

$\left(\text{proj}_{\vec{w}} \vec{v} = (\vec{v} \cdot \vec{w})\, \vec{w} \right.$
where $\vec{w}$ is a unit vector
$\left. \vec{w} \cdot \vec{w} = 1 \,,\ \|\vec{w}\| = 1 \right)$

$$\text{proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

"projection of $\vec{v}$ onto $\vec{w}$"

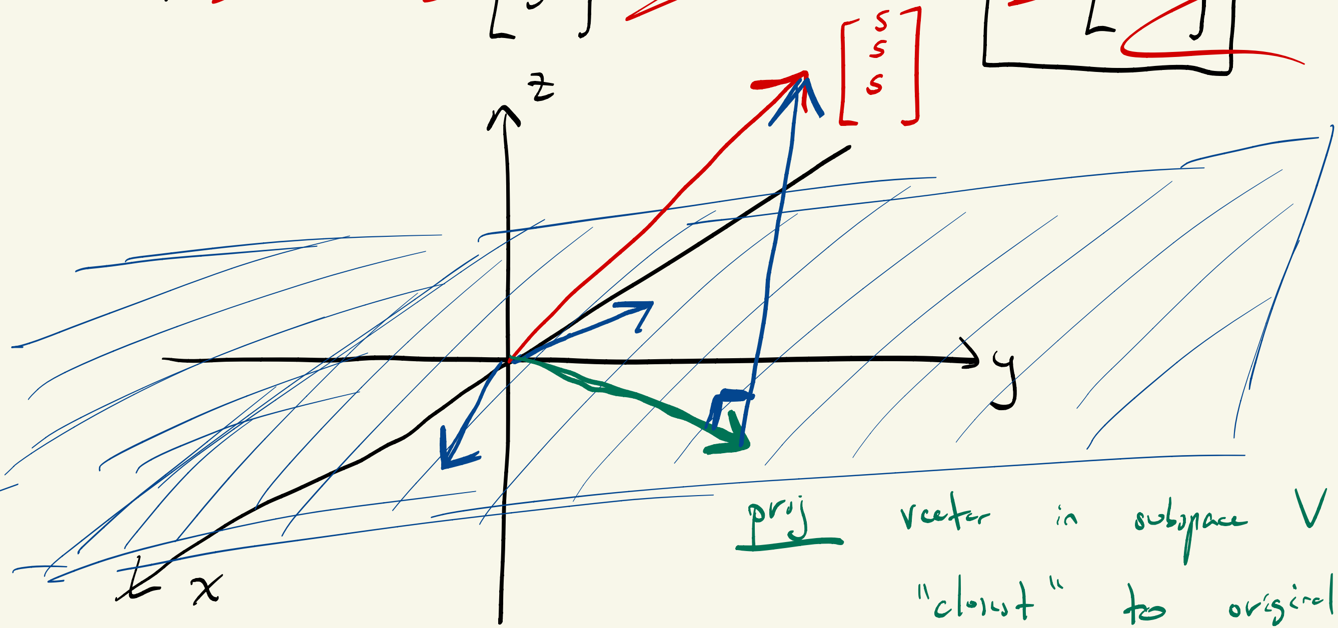
2.3: Find ortho. projection of $\begin{bmatrix} s \\ s \\ s \end{bmatrix}$ onto the subspace

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ \\ 1 \end{bmatrix}$

$\begin{bmatrix} \\ \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \\ 1 \\ 1 \end{bmatrix} = 4 + 1 + 1 = 6$

Note: I did this incorrectly in discussion, Check Worksheet 3 Sol. for correct way

~~The projection of~~ $\cancel{\begin{bmatrix}5\\5\\5\end{bmatrix}}$ ~~onto V is~~ $\cancel{\boxed{\frac{10}{3}\begin{bmatrix}1\\2\\1\end{bmatrix}}}$



<u>proj</u> vector in subspace V "closest" to original vector $\begin{bmatrix}5\\5\\5\end{bmatrix}$

V subspace of $\mathbb{R}^n$

$$V^\perp = \left\{ \vec{w} \in \mathbb{R}^n : \boxed{\vec{w} \cdot \vec{v} = 0} \text{ for } \underline{\text{all}} \ \vec{v} \in V \right\}$$

"orthogonal complement"